The Binomial Expansion Cheat Sheet

The binomial expansion can be used to expand brackets raised to large powers. It can be used to simplify probability models with a large number of trials, such as those used by manufacturers to predict faults.

Pascal's triangle

You can use Pascal's triangle to quickly expand expressions such as $(x+2y)^3$. Consider the expansions of $(a+b)^n$ for n = 0,1,2,3 and 4:

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Each coefficient is the sum of the 2 coefficients immediately above it

Every term in the expansion of $(a+b)^n$ has total index n:
In the $6a^2b^2$ term the total index is $2+2=4$.
In the $4ab^3$ term the total index is $1+3=4$.

Pascal's triangle is formed by adding adjacent pairs of the numbers to find the numbers on the next row.

Here are the first 7 rows of Pascal's triangle:

1
1 + 1
1 + 2 + 1
1 + 3 + 3 + 1
1 + 4 + 6 + 4 + 1
1 + 5 + 10 + 10 + 5 + 1
1 + 6 + 15 + 20 + 15 + 6 + 1

The third row of Pascal's triangle gives the coefficients in the expansion of $(a+b)^2$

The $(n+1)th$ row of Pascal's triangle gives he coefficients in the expansion of $(a+b)^n$.

Example 1:
Use Pascal's triangle to find the expansions of:
a. $(x+2y)^3$ b. $(2x-5)^4$

a. $(x+2y)^3$
The coefficients are 1, 3, 3, 1 so:
$(x+2y)^3 = 1x^3 + 3x^2(2y) + 3x(2y)^2 + 1(2y)^3$
$= x^3 + 6x^2y + 12xy^2 + 8y^3$

Index = 3 so look at the 4th row of Pascal's triangle to find the coefficients.

This is the expansion of $(a+b)^3$ with $a = x$ and $b = 2y$. Use brackets to ensure you don't make a mistake.

b. $(2x-5)^4$
The coefficients are 1, 4, 6, 4, 1 so:
$(2x-5)^4 = 1(2x)^4 + 4(2x)^3(-5) + 6(2x)^2(-5)^2 + 4(2x)(-5)^3 + 1(-5)^4$
$= 16x^4 - 160x^3 + 600x^2 - 1000x + 625$

Index = 4 so look at the 5th row of Pascal's triangle.

This is the expansion of $(a+b)^4$ with $a = 2x$ and $b = -5$

Example 2:
The coefficient of x^2 in the expansion of of $(2-cx)^3$ is 294. Find the possible values of the constant c. (Note: if there is an unknown in the expression, form an equation involving the unknown)

The coefficients are 1, 3, 3, 1:
The term in x^2 is $3 \times 2(-cx)^2 = 6c^2x^2$
So, $6c^2 = 294$
$c^2 = 49 \;\Rightarrow c \pm 7$

Factorial notation

Combinations and factorial notation can help you expand binomial expressions. For larger indices, it is quicker than using Pascal's triangle.

Using factorial notation $3 \times 2 \times 1 = 3!$
You can use factorial notation and your calculator to find entries in Pascal's triangle quickly. The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$:
${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$

Example 3: Calculate
a. 5! b. ${}^5C_2$ c. the 6th entry in the 10th row of Pascal's triangle

a. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
b. ${}^5C_2 = \frac{5!}{2!3!} = \frac{120}{12} = 10$
c. ${}^9C_5 = 126$

Use the ${}^nC_r$ and ! functions on your calculator to answer this question.

You can calculate ${}^5C_2$ by using the ${}^nC_r$ function on your calculator.
${}^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{5!}{2!\,(5-2)!}$

The rth entry in the nth row is ${}^{n-1}C_{r-1}$

The binomial expansion

The binomial expansion is a rule that allows you to expand brackets. You can use $\binom{n}{r}$ to work out the coefficients in the binomial expansion. For example,
in the expansion of $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$, to find the b^3 term you can choose multiples of b from 3 different brackets. You can do this in $\binom{5}{3}$ ways so the b^3 term is $\binom{5}{3}a^2b^3$.

The binomial expansion is:
$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

Example 4 : Use the binomial theorem to find the expansion of $(3-2x)^5$.

$(3-2x)^5 = 3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \binom{5}{3}3^2(-2x)^3 + \binom{5}{4}3^1(-2x)^4 + (-2x)^5$
$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$

There will be 6 terms. Each term has a total index of 5. Use $(a+b)^n$ with $a = 3$, $b = -2x$ and $n = 5$

Solving Binomial Problems

You can use the general term of the binomial expansion to find individual coefficients in a binomial expansion.

In the expansion of $(a+b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

Example 6:
a. Find the coefficient of x^4 in the binomial expansion of $(2+3x)^{10}$.
x^4 term $= \binom{10}{4}2^6(3x)^4$
$= 210 \times 64 \times 81x^4$
$= 1088640x^4$
The coefficient of x^4 in the binomial expansion of $(2+3x)^{10}$ is 1088640.

b. Find the coefficient of x^3 in the binomial expansion of $(2+x)(3-2x)^7$.

$(3-2x)^7$

First, find the first four terms of the binomial expansion of $(3-2x)^7$

$= 3^7 + \binom{7}{1}3^6(-2x) + \binom{7}{2}3^5(-2x)^2 + \binom{7}{3}3^4(-2x)^3 + \cdots$
$= 2187 - 10206x + 20412x^2 - 22680x^3 + \cdots$
$\Rightarrow (2+x)(\,2187 - 10206x + 20412x^2 - 22680x^3 + \cdots)$

Now expand the brackets $(2+x)(3-2x)^7$

x^3 term $= 2 \times (-22680x^3) + x \times 20412x^2$
$= -24948x^3$
The coefficient of x^3 in the binomial expansion of $(2+x)(3-2x)^7$ is -24948.

There are 2 ways of making the x^3 term: (constant term $\times\, x^3$ term) and (x term $\times\, x^2$ term)

Binomial Estimation

If the value of x is less than 1, then x^n gets smaller as n gets larger. If x is small you can sometimes ignore large powers of x to approximate a function or estimate a value.

Example 9:
a. Find the first four terms of the binomial expansion, in ascending powers of x, of $\left(1-\frac{x}{4}\right)^{10}$.

$\left(1-\frac{x}{4}\right)^{10} = 1^{10} + \binom{10}{1}1^9\left(-\frac{x}{4}\right) + \binom{10}{2}1^8\left(-\frac{x}{4}\right)^2 + \binom{10}{3}1^7\left(-\frac{x}{4}\right)^3 + \cdots$
$= 1 - 2.5x + 2.8125x^2 - 1.875x^3 + \cdots$

b. Use your expansion to estimate the value of 0.975^{10}, giving your answer to 4 decimal places.
We want $\left(1-\frac{x}{4}\right) = 0.975$
$\frac{x}{4} = 0.025$
$x = 0.1$

Calculate value of x

Substitute $x = 0.1$ into the expansion for $\left(1-\frac{x}{4}\right)^{10}$ from part a:
$0.975^{10} \approx 1 - 0.25 + 0.028125 - 0.001875$
$= 0.77625$
$0.975^{10} \approx 0.7763$ to 4 d.p

Using a calculator, $0.975^{10} = 0.77632962$. so, approximation is correct.