The Binomial Expansion Cheat Sheet

The binomial expansion can be used to expand brackets raised to large powers. It can be used to simplify probability models with a large number of trials, such as those used by manufacturers to predict faults.

Pascal's triangle

You can use Pascal's triangle to quickly expand expressions such as $(x + 2y)^3$. Consider the expansions of $(a + b)^n$ for n = 0,1,2,3 and 4:

$$(a + b)^{0} = 1$$

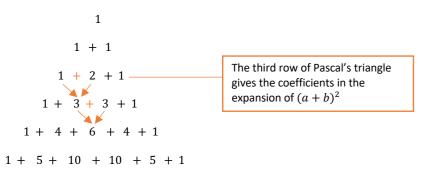
$$(a + b)^{1} = 1a + 1b$$

$$(a + b)^{2} = 1a^{2} + 2ab + 1b^{2}$$

$$(a + b)^{3} = 1a^{3} + 3a^{2}b + 3ab^{2} + 1b^{3}$$
Each coefficient is the sum of the 2 coefficients immediately above it
$$(a + b)^{4} = 1a^{4} + 4a^{3}b + 6a^{2}b^{2} + 4ab^{3} + 1b^{4}$$
Every term in the expansion of $(a + b)^{n}$ has total index n:
In the $6a^{2}b^{2}$ term the total index is $2 + 2 = 4$.
In the $4ab^{3}$ term the total index is $1 + 3 = 4$.

Pascal's triangle is formed by adding adjacent pairs of the numbers to find the numbers on the next row.

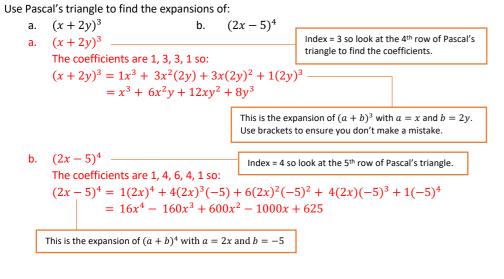
Here are the first 7 rows of Pascal's triangle:



1 + 6 + 15 + 20 + 15 + 6 + 1

The (n + 1)th row of Pascal's triangle gives he coefficients in the expansion of $(a + b)^n$.

Example 1:





Example 2:

The coefficient of x^2 in the expansion of of $(2 - cx)^3$ is 294. Find the possible values of the constant c. (Note: if there is an unknown in the expression, form an equation involving the unknown)

The coefficients are 1, 3, 3, 1:
The term in
$$x^2$$
 is $3 \times 2(-cx)^2 = 6c^2x^2$
So, $6c^2 = 294$
 $c^2 = 49 \implies c \pm 7$

Factorial notation

Combinations and factorial notation can help you expand binomial expressions. For larger indices, it is quicker than using Pascal's triangle.

Using factorial notation
$$3 \times 2 \times 1 = 3!$$

You can use factorial notation and your calculator to find entries in Pascal's triangle quickly. The number of ways of choosing r items from a group of n items is written as ${}^{n}C_{r}$ or $\binom{n}{r}$:

$${}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The *r*th entry in the *n*th row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = {\binom{n-1}{r-1}}$

Example 3: Calculate a. 5!

a. 5! b.
$${}^{5}C_{2}$$
 c. the 6th entry in the 10th row of Pascal's triangle
a. 5! = 5 × 4 × 3 × 2 × 1 = 120
b. ${}^{5}C_{2} = \frac{5!}{2!3!} = \frac{120}{12} = 10$
c. ${}^{9}C_{5} = 126$
The *r*th entry in the *n*th row is ${}^{n-1}C_{r-1}$

The binomial expansion

The binomial expansion is a rule that allows you to expand brackets. You can use $\binom{n}{r}$ to work out the coefficients in the binomial expansion. For example,

in the expansion of $(a + b)^5 = (a + b)(a + b)(a + b)(a + b)(a + b)$, to find the b^3 term you can choose multiples of b from 3 different brackets. You can do this in $\binom{5}{2}$ ways so the b^3 term is $\binom{5}{2}a^2b^3$.

The binomial expansion is:

$$(a+b)^{n} = a^{n} + {\binom{n}{1}}a^{n-1}b + {\binom{n}{2}}a^{n-2}b^{2} + \dots + {\binom{n}{r}}a^{n-r}b^{r} + \dots + b^{n}$$

$$(a+b)^{n} = a^{n} + {\binom{n}{1}}a^{n-1}b + {\binom{n}{2}}a^{n-2}b^{2} + \dots + {\binom{n}{r}}a^{n-r}b^{r} + \dots + b^{n}$$

where $\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$

Example 4 : Use the binomial theorem to find the expansion of $(3 - 2x)^5$.

$$(3 - 2x)^{5} = 3^{5} + {5 \choose 1} 3^{4} (-2x) + {5 \choose 2} 3^{3} (-2x)^{2} + {5 \choose 3} 3^{2} (-2x)^{3} + {5 \choose 4} 3^{1} (-2x)^{4} + (-2x)^{5}$$

= 243 - 810x + 1080x² - 720x³ + 240x⁴ - 32x⁵

There will be 6 terms. Each term has a total index of 5. Use $(a + b)^n$ with a = 3, b = -2x and n = 5

n!

r!(n-r)! = 2!(5-2)!

5!

Solving Binomial Problems

binomial expansion.

Example 6:

a. Find the coeffic
$$x^4$$
 term = $\binom{10}{2^6}$

$$= 210 \times 64$$

The coefficient of
$$x^4$$

$$(3-2x)^7$$

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In the expansion of (a + b)^n the general term is given by \binom{n}{r} a^{n-r} b^r.
                        efficient of x^4 in the binomial expansion of (2 + 3x)^{10}.
               = \binom{10}{4} 2^{6} (3x)^{4}
                =210 \times 64 \times 81x^4
                            )x^4
                            <sup>4</sup> in the binomial expansion of (2 + 3x)^{10} is 1088640.
     b. Find the coefficient of x^3 in the binomial expansion of (2 + x)(3 - 2x)^7.
                                 First, find the first four terms of the binomial
                                 expansion of (3 - 2x)^7
     = 3^{7} + \binom{7}{1} 3^{6} (-2x) + \binom{7}{2} 3^{5} (-2x)^{2} + \binom{7}{2} 3^{4} (-2x)^{3} + \cdots
      = 2187 - 10206x + 20412x^2 - 22680x^3 + \cdots
      \Rightarrow (2+x)(2187 - 10206x + 20412x^2 - 22680x^3 + \cdots)
                              Now expand the brackets (2 + x)(3 - 2x)^7
      x^{3} term = 2 × (-22680x^{3}) + x × 20412x^{2}
                = -24948x^{3}
                                                          There are 2 ways of making the x^3 term:
      The coefficient of x^3 in the binomial
                                                          (constant term \times x^3 term) and (x term
      expansion of (2 + x)(3 - 2x)^7 is -24948.
                                                          \times x^2 term)
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Binomial Estimation

If the value of x is less than 1, then x^n gets smaller as n gets larger. If x is small you can sometimes ignore large powers of x to approximate a function or estimate a value.

a. Find the first fo
$$\left(1-\frac{x}{4}\right)^{10}$$
.

$$(1 x)^{10} - 110$$

$$=1 - 2.5x + 2.8$$

decimal places. We want $(1 - \frac{x}{4}) = 0.975$

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= 0.77625
0.975^{10}\approx 0.7763 to 4 d.p \_\_\_
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You can use the general term of the binomial expansion to find individual coefficients in a

Example 9: a. Find the first four terms of the binomial expansion, in ascending powers of *x*, of

 $\left(1-\frac{x}{4}\right)^{10} = 1^{10} + {\binom{10}{1}}1^9 \left(-\frac{x}{4}\right) + {\binom{10}{2}}1^8 \left(-\frac{x}{4}\right)^2 + {\binom{10}{3}}1^7 \left(-\frac{x}{4}\right)^3 + \cdots$

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8125x^2 - 1.875x^3 + \cdots
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b. Use your expansion to estimate the value of 0.975^{10} , giving your answer to 4

